A NEW SET OF POSTULATES FOR BETWEENNESS, WITH PROOF OF COMPLETE INDEPENDENCE*

BY

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Introduction

The paper on betweenness published by E. V. Huntington and J. R. Kline in 1917 started with a basic list of twelve postulates:

A, B, C, D, 1, 2, 3, 4, 5, 6, 7, 8,

from which eleven sets of independent postulates were selected, as follows:

(1) A, B, C, D, 1, 2;	(5) A, B, C, D, 1, 8;	(9) A, B, C, D, 3, 4, 6;
(2) A, B, C, D, 1, 5;	(6) A, B, C, D, 2, 4;	(10) A, B, C, D, 3, 4, 7;
(3) A, B, C, D, 1, 6;	(7) A, B, C, D, 2, 5;	(11) A, B, C, D, 3, 4, 8.
(4) A, B, C, D, 1, 7;	(8) A, B, C, D, 3, 5;	

Eight of these sets contain six postulates each, and three contain seven postulates each.†

In the present paper a new postulate, called postulate 9, is added to the basic list. This new postulate leads to a twelfth set of independent postulates:

(12) A, B, C, D, 9,

in which the number of postulates is reduced to five. Moreover, the new postulate 9 itself is easier to remember and more convenient to handle than any of the other postulates 1—8.

The addition of this new postulate makes desirable an extension of the discussion of the earlier paper so as to include all thirteen of the basic postulates; and this extension has been made in the present paper.

Finally, the postulates of the new set (12) are shown to be completely independent in the sense of E. H. Moore. (In regard to the other sets, a

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† E. V. Huntington and J. R. Kline, *Sets of independent postulates for betweenness*, these Transactions, vol. 18 (1917), pp. 301-325.

recent paper by Mr. W. E. Van de Walle* has shown that sets (1)—(10) are completely independent, while set (11) is not.)

It is hoped that the material now available on the simple relation of "betweenness," including as it does, 12 sets of postulates with the "complete existential theory" of each set, and no less than 200 demonstrated theorems (116 on deducibility and 84 on non-deducibility), may prove of special interest to students of logic, since it provides the most elaborate known example of an abstract deductive theory.

The basic list of thirteen postulates

The universe of discourse consists of all systems K, R, where K is a class of elements, $A, B, C, \ldots$, and R (ABC) is a triadic relation; among these systems (K, R) we designate as "betweenness" systems those that satisfy the following thirteen conditions, or postulates.

Postulate A. $ABC . \supset . CBA.$

(That is, if ABC is true, then CBA is true.)

Postulate B. $A \neq B . B \neq C . C \neq A : \supset : BAC \smile CAB \smile ABC \smile CBA \smile ACB \smile BCA.$

(That is, if A, B, C are distinct, then *at least one* of the six possible permutations will form a true triad.)

Postulate C. $A \neq X . X \neq Y . Y \neq A : \supset : AXY . AYX . = . 0.$

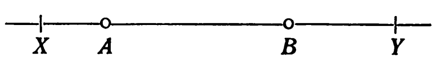
(That is, if A, X, Y are distinct, then we cannot have AXY and AYX both true at the same time.)

Postulate D. $ABC : \supset : A \neq B . B \neq C . C \neq A.$

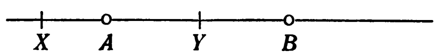
(That is, if ABC is true, then the elements A, B, and C are distinct.)

Postulates 1–8. If A, B, X, Y, are distinct, then:

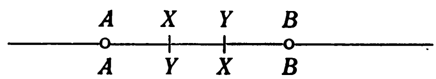
1. $XAB . ABY . \supset . XAY.$



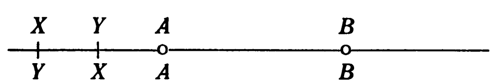
2. $XAB . AYB . \supset . XAY.$
3. $XAB . AYB \ \supset . XYB.$



4. $AXB . AYB . \supset . AXY \smile AYX.$
5. $AXB . AYB . \supset . AXY \smile YXB.$



6. $XAB . YAB . \supset . XYA \smile YXB.$
7. $XAB . YAB . \supset . XYA \smile YXA.$
8. $XAB . YAB . \supset . XYA \smile YXB.$



* W. E. Van de Walle, *On the complete independence of the postulates for betweenness,* in the present number of these Transactions, pp. 249-256.

POSTULATE 9. If A, B, C, X are distinct, then $ABC \,.\, X \,.\supset.\, ABX \smile XBC$.

The new postulate 9 may be read as follows: If ABC is true, and if X is any fourth element distinct from A and B and C, then X must lie either on the right of the middle element (giving ABX), or else on the left of the middle element (giving XBC).

In regard to certain peculiarities of postulates 5 and 8, see under Theorem 5k, below.

THEOREMS ON DEDUCIBILITY

Besides the 71 theorems on deducibility which were stated and proved in the earlier paper, there are found to be 45 new theorems involving the new postulate 9. The proofs of these new theorems are given below, and the complete list of 116 theorems is set forth in Table I′.

The following proofs are supplementary to those given in the earlier paper. In each proof, the number of times that any postulate is used is indicated by an exponent.

THEOREM 1e. *Proof of* 1 *from* A, C, 9.

To prove: $XAB \,.\, ABY \,.\supset.\, XAY$. By A, $ABY \,.\supset.\, YBA$. By 9, $XAB \,.\, Y \,.\supset.\, XAY \smile YAB$. But YAB conflicts with YBA, by C. Hence XAY.

THEOREM 2j. *Proof of* 2 *from* A, C, 9.

To prove: $XAB \,.\, AYB \,.\supset.\, XAY$. By A, $AYB \,.\supset.\, BYA$. By 9, $XAB \,.\, Y \,.\supset.\, XAY \smile YAB$. But if YAB, then by A, BAY, which conflicts with BYA, by C. Hence XAY.

THEOREM 3f. *Proof of* 3 *from* A, C^2, 9^2.

To prove: $XAB \,.\, AYB \,.\supset.\, XYB$. Suppose XYB is false. First, by 9, $AYB \,.\, X \,.\supset.\, AYX \smile XYB$; hence AYX, whence by A, XYA. Second, by 9, $XAB \,.\, Y \,.\supset.\, XAY \smile YAB$; but YAB conflicts with AYB, by A and C; hence XAY. But thirdly, XYA and XAY conflict with each other, by C. Therefore XYB must be true.

THEOREM 3g. *Proof of* 3 *from* A, 1^2, 9^2.

To prove: $XAB \,.\, AYB \,.\supset.\, XYB$. Suppose XYB is false. By 9, $AYB \,.\, X \,.\supset.\, AYX \smile XYB$; hence AYX, whence, by A, XYA. By 9, $XAB \,.\, Y \,.\supset.\, XAY \smile YAB$.

Case 1. If YAB, then by 1, $XYA \,.\, YAB \,.\supset.\, XYB$.

Case 2. If XAY, then by A and 1, $BYA \,.\, YAX \,.\supset.\, BYX$, whence, by A, XYB.

THEOREM 3h. *Proof of* 3 *from* A, B, 2^4, 9.

To prove: $XAB . AYB . \supset . XYB$. Suppose XYB is false. Then, by B and A, $YXB \smile XBY$.

Case 1. If YXB, then by 2, $YXB . XAB . \supset . YXA$; hence, by 2 and A, $BYA . YXA . \supset . BYX$, whence, by A, XYB.

Case 2. If XBY, then by 2 and A, $YBX . BAX . \supset . YBA$.

Now by 9, $AYB . X . \supset . AYX \smile XYB$; but XYB is false; hence AYX; whence, by A, XYA. Then by 2, $XYA . YBA . \supset . XYB$.

THEOREM 3i. *Proof of* 3 *from* A, 2^3, 6, 9.

To prove: $XAB . AYB . \supset . XYB$. Suppose XYB is false. By 9, $AYB . X . \supset . AYX \smile XYB$; hence AYX. Then by A and 2, $BAX . AYX . \supset . BAY$, whence, by A, YAB. Then by 6, $XAB . YAB . \supset . XYB \smile YXB$; but XYB is false; hence YXB. Then by 2, $YXB . XAB . \supset . YXA$. Hence, by A and 2, $BYA . YXA . \supset . BYX$, whence, by A, XYB.

THEOREM 3j. *Proof of* 3 *from* A, 2^4, 7, 9.

To prove: $XAB . AYB . \supset . XYB$. Suppose XYB is false. By 9, $AYB . X . \supset . AYX \smile XYB$; hence AYX, whence, by A, XYA. Then by A and 7, $BYA . XYA . \supset . BXY \smile XBY$.

Case 1. If BXY, then by A and 2, $YXB . XAB . \supset . YXA$. Then by A and 2, $BYA . YXA . \supset . BYX$, whence by A, XYB.

Case 2. If XBY, then by A and 2, $YBX . BAX . \supset . YBA$. Then by 2, $XYA . YBA . \supset . XYB$.

THEOREM 3k. *Proof of* 3 *from* A, 2^2, 8, 9.

To prove: $XAB . AYB . \supset . XYB$. Suppose XYB is false. By 9, $AYB . X . \supset . AYX \smile XYB$; hence AYX, whence by A, XYA. By 2, $XAB . AYB . \supset . XAY$, whence by A, YAX. Then by A and 8, $YAX . BAX . \supset . YBA \smile BYX$, whence by A, $YBA \smile XYB$; but XYB is false; hence YBA. Then by 2, $XYA . YBA . \supset . XYB$.

THEOREM 4k. *Proof of* 4 *from* A, C, 9^2.

To prove: $AXB . AYB . \supset . AXY \smile AYX$. Suppose both AXY and AYX are false.

By 9, $AXB . Y . \supset . AXY \smile YXB$; hence YXB.

By 9, $AYB . X . \supset . AYX \smile XYB$; hence XYB.

But YXB and XYB conflict with each other, by A and C. Hence $AXY \smile AYX$.

THEOREM 4l. *Proof of* 4 *from* A, 1^2, 7^2, 9.

To prove: $AXB . AYB . \supset . AXY \smile AYX$. Suppose both AXY and AYX are false. By 9, $AXB . Y . \supset . AXY \smile YXB$; hence YXB.

Then by 7, $AXB . YXB . \supset . AYX \smile YAX$; hence YAX.
By 1, $YAX . AXB . \supset . YAB$.
By 1 and A, $XAY . AYB . \supset . XAB$.
Then by 7, $YAB . XAB . \supset . YXA \smile XYA$. Hence by A, $AXY \smile AYX$.

THEOREM 4m. *Proof of* 4 *from* A, 3^2, 7^2, 9^2.
To prove: $AXB . AYB . \supset . AXY \smile AYX$. Suppose both AXY and AYX are false.
By 9, $AXB . Y . \supset . AXY \smile YXB$. Hence YXB, whence, by A, BXY.
By 9, $AYB . X . \supset . AYX \smile XYB$. Hence XYB, whence, by A, BYX.
Then by 7, $AXB . YXB . \supset . AYX \smile YAX$; hence YAX, whence, by A, XAY.
By 3, $BYX . YAX . \supset . BAX$; and by 3, $BXY . XAY . \supset . BAY$.
Then by 7 and A, $YAB . XAB . \supset . YXA \smile XYA$, whence, by A, $AXY \smile AYX$.

THEOREM 4n. *Proof of* 4 *from* A, 7, 8^2, 9^2.
To prove: $AXB . AYB . \supset . AXY \smile AYX$. Suppose both AXY and AYX are false.
By 9, $AXB . Y . \supset . AXY \smile YXB$; hence YXB. By 9, $AYB . X . \supset . AYX \smile XYB$; hence XYB.
Then by 8, $AXB . YXB . \supset . AYX \smile YAB$; hence YAB.
Also, by 8, $AYB . XYB . \supset . AXY \smile XAB$; hence XAB.
Then by 7, $YAB . XAB . \supset . YXA \smile XYA$, whence, by A, $AXY \smile AYX$.

THEOREM 4o. *Proof of* 4 *from* C^2, 7^3, 8^2, 9^2.
To prove: $AXB . AYB . \supset . AXY \smile AYX$. Suppose both AXY and AYX are false.
By 9, $AXB . Y . \supset . AXY \smile YXB$; hence YXB.
By 9, $AYB . X . \supset . AYX \smile XYB$; hence XYB.
Then by 7, $AXB . YXB . \supset . AYX \smile YAX$; hence YAX. And by 7, $AYB . XYB . \supset . AXY \smile XAY$; hence XAY.
Also by 8, $AXB . YXB . \supset . AYX \smile YAB$; hence YAB and by 8, $AYB . XYB . \supset . AXY \smile XAB$; hence XAB.
Then by 7, $YAB . XAB . \supset . YXA \smile XYA$.
But YXA conflicts with YAX, by C, and XYA conflicts with XAY, by C.
Therefore $AXY \smile AYX$.

THEOREM 4p. *Proof of* 4 *from* 2^2, 9^2.
To prove: $AXB . AYB . \supset . AXY \smile AYX$. Suppose both AXY and AYX are false.
By 9, $AXB . Y . \supset . AXY \smile YXB$; hence YXB.
By 9, $AYB . X . \supset . AYX \smile XYB$; hence XYB.

Then by 2, $AYB.YXB.\supset.AYX$, and by 2, $AXB.XYB.\supset.AXY$. Therefore $AXY\smile AYX$.

THEOREM 5k. *Proof of* 5 *from* 9.

To prove: $AXB.AYB.\supset.AXY\smile YXB$.

By 9, $AXB.Y.\supset.AXY\smile YXB$; which was to be proved. It will be observed that only the first part of the hypothesis is used in the proof. Postulate 9 is "stronger" than postulate 5. By interchanging X and Y, postulate 5 may also be written in the form

$$AYB.AXB.\supset.AYX\smile XYB;$$

which may be proved as follows: By 9,

$$AYB.X.\supset.AYX\smile XYB;$$

which was to be proved. Here again, only the first part of the hypothesis is used in the proof.

Furthermore, since "AXB and AYB" is logically equivalent to "AYB and AXB", postulate 5 may be written in either of the following forms:

$$AXB.AYB.\supset.AYX\smile XYB.$$
$$AYB.AXB.\supset.AXY\smile YXB.$$

It is interesting to notice, however, that no one of these four forms is a significant statement, unless one part of the hypothesis is recognized specifically as the "first part" and the other as the "second part" — a distinction which, strictly speaking, introduces a foreign element into the statement of the proposition.

In order to avoid the necessity of making this arbitrary distinction between the "first" and the "second" term of a pair connected by a simple "and", we may restate postulate 5 in the following less objectionable form:

$$5'.\quad AXB.AYB:\supset:(AXY\smile YXB).(AYX\smile XYB).$$

This should not be regarded as merely a combination of two of the separate statements mentioned above, since, without employing the distinction between "first" and "second", we cannot tell which part of the hypothesis is supposed to be paired with which part of the conclusion. It is only when the statement (5′) is taken as a whole that it can be translated into significant words, without using the distinction between the "first" and "second" parts of the simple conjunction which forms the hypothesis.

Thus, 5′ may be read as follows: "The two triads in the hypothesis contain the same initial element, A, and the same terminal element, B, but different middle elements, X and Y (which we may call the "odd elements"). The conclusion also consists of two parts. One part says that at least one of the following triads is true:

(A) (one odd) (the other odd) or (the other odd) (the one odd) (B); the other part says that at least one of the following is true:

(A) (the other odd) (the one odd) or (the one odd) (the other odd) (B)."

Now neither of these parts alone gives us any definite information unless we are able to recognize the "one" as X and the "other" as Y (or vice versa); but the two parts together give an unequivocal conclusion whether the "one" $= X$ and the "other" $= Y$, or the "one" $= Y$ and the "other" $= X$.

Precisely the same remarks apply to postulate 8, which may be re-stated more strictly as follows:

$$8'.\quad XAB\,.\,YAB:\supset:(XYA \smile YXB)\,.\,(YXA \smile XYB).$$

Fortunately, these logical refinements do not affect the essential reasoning, provided the precaution already stated in the footnote on page 318 of the earlier paper is observed.

THEOREM 6k. *Proof of* 6 *from* A, B, C^2, 9.

To prove: $XAY.YAB.\supset.XYB \smile YXB$. By B and A, $XYB \smile YXB \smile XBY$. Suppose XBY. Then by 9, $XBY.A.\supset.XBA \smile ABY$. But XBA conflicts with XAB, by C; and $ABY.\supset.YBA$, by A, which conflicts with YAB, by C. Hence $XYB \smile YXB$.

THEOREM 6l. *Proof of* 6 *from* A, C, 7, 9.

To prove: $XAB.YAB.\supset.XYB \smile YXB$.

By 7, $XAB.YAB.\supset.XYA \smile YXA$.

Case 1. If XYA, then by 9, $XYA.B.\supset.XYB \smile BYA$. But BYA conflicts with YAB, by C and A. Hence, in Case 1, XYB.

Case 2. If YXA, then by 9, $YXA.B.\supset.YXB \smile BXA$. But BXA conflicts with XAB, by C and A. Hence, in Case 2, YXB.

THEOREM 6m. *Proof of* 6 *from* A, 2^4, 7^3, 9^2.

To prove: $XAB.YAB.\supset.XYB \smile YXB$. Suppose both XYB and YXB (and hence, by A, also BYX and BXY) are false.

By 7, $XAB.YAB.\supset.XYA \smile YXA$.

Case 1. If XYA, then by 9, $XYA.B.\supset.XYB \smile BYA$. But XYB is false. Hence BYA, and by A, AYB. Then by 7, $BYA.XYA.\supset.BXY \smile XBY$.

But BXY is false. Hence XBY. Then by 2 and A, $YBX . BAX . \supset . YBA$. Hence by 2, $XYA . YBA . \supset . XYB$.

Case 2. If YXA, then by 9, $YXA . B . \supset . YXB . BXA$. But YXB is false. Hence BXA. Then by 7 and A, $BXA . YXA . \supset . BYX \smile YBX$. But BYX is false. Hence YBX. Then by 2 and A, $XBY . BAY . \supset . XBA$. Hence by 2, $YXA . XBA . \supset . YXB$.

Therefore $XYB \smile YXB$.

THEOREM 6n. *Proof of* 6 *from* A, 2, 8^3, 9.

To prove: $XAB . YAB . \supset . XYB \smile YXB$. Suppose both XYB and YXB are false. Then, by A, both BYX and BXY are false.

By 8, $XAB . YAB . \supset . XYA \smile YXB$; hence XYA.

By 8, $YAB . XAB . \supset . YXA \smile XYB$; hence YXA.

By 9, $XYA . B . \supset . XYB \smile BYA$; hence BYA.

By 8, $BYA . XYA . \supset . BXY \smile XBA$; hence XBA.

By 2, $YXA . XBA . \supset . YXB$ contrary to supposition.

THEOREM 6o. *Proof of* 6 *from* A, 4, 8^2, 9^2.

To prove: $XAB . YAB . \supset . XYB \smile YXB$. Suppose both XYB and YXB are false.

By 8, $XAB . YAB . \supset . XYA \smile YXB$; hence XYA, and by A, AYX.

By 8, $YAB . XAB . \supset . YXA \smile XYB$; hence YXA, and by A, AXY.

By 9, $XYA . B . \supset . XYB \smile BYA$; hence BYA.

By 9, $YXA . B . \supset . YXB \smile BXA$; hence BXA.

Then by 4, $BXA . BYA . \supset . BXY \smile BYX$. Hence, by A, $XYB \smile BXY$.

THEOREM 6p. *Proof of* 6 *from* A, 7, 8^4, 9^2.

To prove: $XAB . YAB . \supset . XYB \smile YXB$. Suppose both XYB and YXB are false. Then by A, BYX and BXY are false.

By 8, $XAB . YAB . \supset . XYA \smile YXB$; hence XYA.

By 8, $YAB . XAB . \supset . YXA \smile XYB$; hence YXA.

By 9, $XYA . B . \supset . XYB \smile BYA$; hence BYA.

By 9, $YXA . B . \supset . YXB \smile BXA$; hence BXA.

By 8, $BYA . XYA . \supset . BXY \smile XBA$; hence XBA.

By 8, $BXA . YXA . \supset . BYX \smile YBA$; hence YBA.

Then by 7, $XBA . YBA . \supset . XYB \smile YXB$.

THEOREM 6q. *Proof of* 6 *from* C^3, 8^4, 9^2.

To prove: $XAB . YAB . \supset . XYB \smile YXB$. Suppose both XYB and YXB are false.

By 8, $XAB . YAB . \supset . XYA \smile YXB$; hence XYA.

By 8, $YAB . XAB . \supset . YXA \smile XYB$; hence YXA.

By 9, $XYA . B . \supset . XYB \smile BYA$; hence BYA.

By 9, $YXA . B . \supset . YXB \smile BXA$; hence BXA.

By 8, $BYA . XYA . \supset . BXY \smile XBA$. But XBA conflicts with XAB by C. Hence BXY.

By 8, $BXA . YXA . \supset . BYX \smile YBA$. But YBA conflicts with YAB by C. Hence BYX.

Now BXY and BYX conflict with each other, by C.

Therefore $XYB \smile YXB$.

THEOREM 7k. *Proof of* 7 *from* A, B, C^4, 9^3.

To prove: $XAB . YAB . \supset . XYA \smile YXA$.

By B, $XYB \smile YXB \smile XBY$.

Case 1. If XYB, then by 9, $XYB . A . \supset . XYA \smile AYB$. But AYB conflicts with BAY, by C and A. Hence in Case 1, XYA.

Case 2. If YXB, then by 9, $YXB . A . \supset . YXA \smile AXB$. But AXB conflicts with XAB, by C and A. Hence in Case 2, YXA.

Case 3. Suppose XBY. Then by 9, $XBY . A . \supset . XBA \smile ABY$. But XBA conflicts with XAB, by C; and ABY conflicts with YAB, by C and A. Hence Case 3 is impossible.

Therefore $XYA \smile YXA$.

THEOREM 7l. *Proof of* 7 *from* A, C^2, 6, 9^2.

To prove: $XAB . YAB . \supset . XYA \smile YXA$.

By 6, $XAB . YAB . \supset . XYB \smile YXB$.

Case 1. If XYB, then by 9, $XYB . A . \supset . XYA \smile AYB$. But AYB conflicts with YAB, by C and A. Hence, in Case 1, XYA.

Case 2. If YXB, then by 9, $YXB . A . \supset YXA \smile AXB$. But AXB conflicts with XAB, by C and A. Hence, in Case 2, YXA.

Therefore $XYA \smile YXA$.

THEOREM 7m. *Proof of* 7 *from* A, 4, 8^2, 9^2.

To prove: $XAB . YAB . \supset . XYA \smile YXA$. Suppose both XYA and YXA are false.

By 8, $XAB . YAB . \supset . XYA \smile YXB$; hence YXB.

By 8, $YAB . XAB . \supset . YXA \smile XYB$; hence XYB.

Then by 9, $YXB . A . \supset . YXA \smile AXB$; hence AXB.

And by 9, $XYB . A . \supset . XYA \smile AYB$; hence AYB.

Then by 4, $AXB . AYB . \supset . AXY \smile AYX$. Hence by A, $XYA \smile YXA$.

THEOREM 7n. *Proof of* 7 *from* A, 1^2, 4^3, 6^3, 9^2.

To prove: $XAB . YAB . \supset . XYA \smile YXA$. Suppose both XYA and YXA are false. Then by A, AYX and AXY are false.

By 6, $XAB . YAB . \supset . XYB \smile YXB$.

Case 1. If XYB is true and YXB false, then by 4, $XAB . XYB . \supset . XAY \smile XYA$; hence XAY. Then by 6 and A, $XAY . BAY . \supset . XBY \smile BXY$, whence, by A, $YBX \smile YXB$. But YXB is false, hence YBX. Now by 9, $XYB . A . \supset . XYA \smile AYB$; hence AYB. Then by 1, $AYB . YBX . \supset . AYX$, which is false.

Case 2. If YXB is true and XYB false, then by 4, $YXB . YAB . \supset . YXA \smile YAX$; hence YAX. Then by 6 and A, $YAX . BAX . \supset . YBX \smile BYX$, whence, by A, $XBY \smile XYB$. But XYB is false; hence XBY. By 9, $YXB . A . \supset . YXA \smile AXB$; hence AXB. Then by 1, $AXB . XBY . \supset . AXY$, which is false.

Case 3. If XYB and YXB are both true, then by 9, $XYB . A . \supset . XYA \smile AYB$; hence AYB. And by 9, $YXB . A . \supset . YXA \smile AXB$; hence AXB. Then by 4, $AXB . AYB . \supset . AXY \smile AYX$. Therefore by A, $XYA \smile YXA$.

THEOREM 8n. *Proof of* 8 *from* A, B, C^3, 9^2.

To prove: $XAB . YAB . \supset . XYA \smile YXB$. By B, $XBY \smile XYB \smile YXB$.

Case 1. Suppose XBY; then by 9, $XBY . A . \supset . XBA \smile ABY$. But XBA conflicts with XAB, by C; and ABY conflicts with YAB, by A and C.

Case 2. If XYB, then by 9, $XYB . A . \supset . XYA \smile AYB$. But AYB conflicts with BAY, by A and C.

Therefore $XYA \smile YXB$.

THEOREM 8o. *Proof of* 8 *from* A, B, 1^2, 6^2, 9.

To prove: $XAB . YAB . \supset . XYA \smile YXB$. Suppose both XYA and YXB are false. By 6, $XAB . YAB . \supset . XYB \smile YXB$; hence XYB. Then by 9, $XYB . A . \supset . XYA \smile AYB$; hence AYB.

By B and A, $YXA \smile XAY \smile XYA$; hence YXA or XAY. But if YXA, then by 1, $YXA . XAB . \supset . YXB$, which is false; hence XAY. Then by 6 and A, $XAY . BAY . \supset . XBY \smile BXY$. But if BXY, then by A, YXB, which is false; hence XBY, whence, by A, YBX. Then by 1, $AYB . YBX . \supset . AYX$, whence, by A, XYA, which is false.

Therefore $XYA \smile YXB$.

THEOREM 8p. *Proof of* 8 *from* A, C, 6, 9.

To prove: $XAB . YAB . \supset . XYA \smile YXB$. Suppose both XYA and YXB are false. By 6, $XAB . YAB . \supset . XYB \smile YXB$; hence XYB. Then by 9, $XYB . A . \supset . XYA \smile AYB$. But AYB conflicts with YAB by C and A; and XYA is false.

Therefore $XYA \smile YXB$.

THEOREM 8q. *Proof of* 8 *from* A, C, 7, 9.

To prove: $XAB . YAB . \supset . XYA \smile YXB$. We may vary the method of proof, as follows: By 7, $XAB . YAB . \supset . XYA \smile YXA$. If XYA, the theorem is established. Suppose YXA; then by 9, $YXA.B.\supset.YXB \smile BXA$. If YXB, the theorem is established. Suppose BXA. By A, $XAB.\supset.BAX$, which conflicts with BXA, by C. Therefore the theorem must be true.

THEOREM 8r. *Proof of* 8 *from* A, 1, 4, 6^2, 9.

To prove: $XAB.YAB.\supset.XYA \smile YXB$. Suppose both XYA and YXB are false. By 6, $XAB.YAB.\supset.XYB \smile YXB$; hence XYB. Then by 4, $XYB.XAB.\supset.XYA \smile XAY$; hence XAY; and by 9, $XYB.A.\supset.XYA \smile AYB$; hence AYB.

By A and 6, $BAY.XAY.\supset.BXY \smile XBY$, whence, by A, $YXB \smile YBX$; hence YBX. Then by 1, $AYB.YBX.\supset.AYX$, whence, by A, XYA, which is false.

Therefore $XYA \smile YXB$.

THEOREM 8s. *Proof of* 8 *from* A, 2^3, 7^2, 9.

To prove: $XAB.YAB.\supset.XYA \smile YXB$. Suppose both XYA and YXB are false. By 7, $XAB.YAB.\supset.XYA \smile YXA$; hence YXA. Then by 9, $YXA.B.\supset.YXB \smile BXA$; hence BXA. Then by 7, $YXA.BXA.\supset.YBX \smile BYX$.

Case 1. If YBX, then by A and 2, $XBY.BAY.\supset.XBA$. Then by 2, $YXA.XBA.\supset.YXB$.

Case 2. If BYX, then by A and 2, $XYB.YAB.\supset.XYA$.

Therefore $XYA \smile YXB$.

THEOREM 9a. *Proof of* 9 *from* A, B, 1, 2.

To prove: $ABC.X.\supset.ABX \smile XBC$. By B and A we have $XBC \smile BCX \smile BXC$. If BCX, then $ABC.BCX.\supset.ABX$, by 1. If BXC, then $ABC.BXC.\supset.ABX$, by 2. Hence $ABX \smile XBC$.

THEOREM 9b. *Proof of* 9 *from* A, B^3, C^2, 1^4, 5.

To prove: $ABC.X.\supset.ABX \smile XBC$. Suppose both ABX and XBC are false. Then, by B and A, $AXB \smile BAX$ and by B and A, $BXC \smile BCX$.

Case 1. If BCX, then by 1, $ABC.BCX.\supset.ABX$.

Case 2. If BAX, then by 1 and A, $CBA.BAX.\supset.CBX$, whence, by A, XBC.

Case 3. If BXC and AXB, use B again: $ACX \smile CAX \smile CXA$.

Suppose ACX. Then by 1 and A, $ACX.CXB.\supset.ACB$, contrary to ABC, by C.

Suppose CAX. Then by 1, $CAX.AXB.\supset.CAB$, contrary to ABC, by C and A.

Suppose CXA. Then by 5 and A, $CBA.CXA.\supset.CBX \smile XBA$. Hence, by A, $ABX \smile XBC$.

THEOREM 9c. *Proof of* 9 *from* A, B^2, C^4, 1^2, 6^2.

To prove: $ABC.X. \supset .ABX \smile XBC$.

Suppose both ABX and XBC are false. Then, by B and A, $AXB \smile BAX$ and, by B and A, $BXC \smile BCX$.

Case 1. If BAX and BXC, then by A and 1, $XAB.ABC. \supset .XAC$, whence by A and 6, $BAX.CAX. \supset .BCX \smile CBX$; but both BCX and CBX conflict with BXC, by C and A.

Case 2. If AXB and BXC, then, by A and 6, $AXB.CXB. \supset .ACB \smile CAB$, both of which conflict with ABC, by C and A.

Case 3. If BCX, then, by 1, $ABC.BCX. \supset .ABX$, which is false.

Therefore $ABX \smile XBC$.

THEOREM 9d. *Proof of* 9 *from* A, B^2, C^2, 1^4, 7.

To prove: $ABC.X. \supset .ABX \smile XBC$. Suppose both ABX and XBC are false. Then, by B and A, $BAX \smile AXB$, and, by B and A, $BCX \smile CXB$.

Case 1. If BAX, then by 1 and A, $CBA.BAX. \supset .CBX$, whence XBC, by A.

Case 2. If BCX, then by 1, $ABC.BCX. \supset .ABX$.

Case 3. If CXB and AXB, then by 7, $CXB.AXB. \supset .CAX \smile ACX$. But if CAX, then by 1, $CAX.AXB. \supset .CAB$, and if ACX, then by 1, $ACX.CXB. \supset .ACB$, both of which conflict with ABC, by C.

Therefore $ABX \smile XBC$.

THEOREM 9e. *Proof of* 9 *from* A, B^2, C^3, 1^2, 8^2.

To prove: $ABC.X. \supset .ABX \smile XBC$. Suppose both ABX and XBC were false. Then, by B and A, $AXB \smile BAX$, and, by B and A, $CXB \smile BCX$.

Case 1. If BCX, then, by 1, $ABC.BCX. \supset .ABX$.

Case 2. If BAX, then, by 1 und A, $CBA.BAX. \supset .CBX$, whence, by A, XBC.

Case 3. Suppose AXB and CXB, then, by 8, $AXB.CXB. \supset .ACX \smile CAB$ and, by 8, $CXB.AXB. \supset .CAX \smile ACB$. But ACX and CAX conflict with each other, by A and C; CAB conflicts with ABC, by A and C; and ACB conflicts with ABC, by C. Hence Case 3 is impossible.

Therefore $ABX \smile XBC$.

THEOREM 9f. *Proof of* 9 *from* A, B^2, C^2, 2^2, 4.

To prove: $ABC.X. \supset .ABX \smile XBC$. Suppose both ABX and XBC were false. Then, by B and A, $BXA \smile BAX$, and, by B and A, $BXC \smile BCX$.

Case 1. If BXC, then, by 2, $ABC.BXC. \supset .ABX$.

Case 2. If BXA, then, by 2 and A, $CBA.BXA. \supset .CBX$, whence, by A, XBC.

Case 3. If BAX and BCX, then, by 4, $BAX \,.\, BCX \,.\supset .\, BAC \smile BCA$, both of which conflict with ABC, by C.

Hence $ABX \smile XBC$.

THEOREM 9g. *Proof of* 9 *from* A, B², C³, 2², 5².

To prove: $ABC \,.\, X \,.\supset .\, ABX \smile XBC$. Suppose both ABX and XBC were false. Then, by B and A, $BAX \smile BXA$ and $BCX \smile BXC$.

Case 1. If BXC, then, by 2, $ABC \,.\, BXC \,.\supset .\, ABX$.

Case 2. If BXA, then, by 2 and A, $CBA \,.\, BXA \,.\supset .\, CBX$, whence, by A, XBC.

Case 3. If BAX, and BCX, then, by 5, $BAX \,.\, BCX \,.\supset .\, BAC \smile CAX$, and, by 5, $BCX \,.\, BAX \,.\supset .\, BCA \smile ACX$. But BAC conflicts with ABC, by C and A; BCA conflicts with ABC, by C and A; and CAX and ACX conflict with each other, by C and A. Hence $ABX \smile XBC$.

THEOREM 9h. *Proof of* 9 *from* A, B, 3², 5.

To prove: $ABC \,.\, X \,.\supset .\, ABX \smile XBC$. By A, CBA. By B and A, $XAC \smile XCA \smile AXC$.

If XAC, then, by 3, $XAC \,.\, ABC \,.\supset .\, XBC$.

If XCA, then, by 3, $XCA \,.\, CBA \,.\supset .\, XBA$, whence ABX by A.

If AXC, then, by 5, $ABC \,.\, AXC \,.\supset .\, ABX \smile XBC$.

Hence, in any case, $ABX \smile XBC$.

THEOREM 9i. *Proof of* 9 *from* A, B², C², 3², 4, 6.

To prove: $ABX \,.\, X \,.\supset .\, ABX \smile XBC$. Suppose both ABX and XBC were false. Then by B and A, $BAX \smile AXB$ and $BCX \smile CXB$.

Case 1. Suppose BAX and BCX. Then, by 4, $BAX \,.\, BCX \,.\supset .\, BAC \smile BCA$.

Case 2. Suppose BAX and CXB. Then, by 3 and A, $CXB \,.\, XAB \,.\supset .\, CAB$.

Case 3. Suppose AXB and BCX. Then, by 3 and A, $AXB \,.\, XCB \,.\supset .\, ACB$.

Case 4. Suppose AXB and CXB. Then, by 6, $AXB \,.\, CXB \,.\supset .\, ACB \smile CAB$.

Hence, in any case, by A, $CAB \smile ACB$, both of which conflict with ABC, by C and A. Therefore $ABX \smile XBC$.

THEOREM 9j. *Proof of* 9 *from* A, B, 3², 4², 7.

To prove: $ABC \,.\, X \,.\supset .\, ABX \smile XBC$.

By A, CBA. By B, $AXC \smile XAC \smile XCA$.

Case 1. If AXC, then also, by A, CXA. Then by 4, $AXC \,.\, ABC \,.\supset .\, AXB \smile ABX$, and also, by 4, $CXA \,.\, CBA \,.\supset .\, CXB \smile CBX$, whence, by A, $CXB \smile XBC$. Then if ABX and XBC are both false, we must have AXB and CXB. Hence, by 7, $CXB \,.\, AXB \,.\supset .\, CAX \smile ACX$, whence, by A, $XAC \smile XCA$.

Case 2. If XAC, then, by 3, $XAC \,.\, ABC \,.\supset .\, XBC$.

Case 3. If XCA, then, by 3, $XCA . CBA . \supset . XBA$, whence by A, ABX.

Therefore, in any case, $ABX \smile XBC$.

THEOREM 9k. *Proof of* 9 *from* A, B^2, C^3, 3^2, 4, 8^2.

Proof same as for Theorem 9i down to

Case 4. Suppose AXB and CXB. Then, by 8, $AXB . CXB . \supset . ACX \smile CAB$, and by 8, $CXB . AXB . \supset . CAX \smile ACB$. But CAB and ACB conflict with ABC, by C and A, and ACX and CAX conflict with each other, by C and A; so that Case 4 is impossible. Also, Cases 1, 2, and 3 conflict with ABC, by C and A.

Hence $ABX \smile XBC$ must be true.

THEOREM 9l. *Proof of* 9 *from* A, B, 2, 3^2, 4.

To prove: $ABC . X . \supset . ABX \smile XBC$.

By B, $XAC \smile XCA \smile AXC$.

Case 1. If XAC, then by 3, $XAC . ABC . \supset . XBC$.

Case 2. If XCA, then by 3 and A, $XCA . CBA . \supset . XBA$, whence, by A, ABX.

Case 3. If AXC, then by 4, $ABC . AXC . \supset . ABX \smile AXB$. But if AXB, then by 2 and A, $CBA . BXA . \supset . CBX$, whence, by A, XBC.

Therefore, $ABX \smile XBC$.

THEOREM 9m. *Proof of* 9 *from* A, B^2, 1, 3^2, 7.

To prove: $ABC . X . \supset . ABX \smile XBC$. Suppose both ABX and XBC are false.

By B, $XAC \smile XCA \smile AXC$. But if XAC, then by 3, $XAC . ABC . \supset . XBC$, which is false; and if XCA, then by 3 and A, $XCA . CBA . \supset . XBA$, whence, by A, ABX, which is false. Therefore AXC.

Now by B, $XBC \smile BCX \smile CXB$. But XBC is false; and if BCX, then by 1, $ABC . BCX . \supset . ABX$, which is false. Therefore CXB, whence, by A, BXC.

Then by 7, $BXC . AXC . \supset . BAX \smile ABX$. But ABX is false; and if BAX, then by 1 and A, $CBA . BAX . \supset . CBX$, whence, by A, XBC, which is false.

Therefore $ABX \smile XBC$.

These 45 new theorems, together with the 71 theorems proved in the earlier paper, complete the list of 116 theorems on deducibility among the thirteen postulates of our revised basic list.

The results are collected for reference in Table I′.

TABLE I′. 116 THEOREMS ON DEDUCIBILITY

Theorem	Postulate	follows from	Set in which used
1a	1	A B C 2 4	6
1b	1	A B C 3 4	9, 10, 11
1c	1	A B C 2 5	7
1d	1	A B C 3 5	8
1e	1	A C 9	12
2a	2	A B C 1 7	4
2b	2	A B C 1 6	3
2c	2	A B C 3 6	9
2d	2	A C 3 7	10
2e	2	A C 3 4 6	9
2f	2	A B C 1 8	5
2g	2	A B C 1 5	2
2h	2	A C 3 8	11
2i	2	A C 3 5	8
2j	2	A C 9	12
3a	3	A B C 1	1, 2, 3, 4, 5
3b	3	A B C 2	1, 6, 7
3c	3	A C 2 6	
3d	3	A 1 2	1
3e	3	A C 2 8	
3f	3	A C 9	12
3g	3	A 1 9	
3h	3	A B 2 9	
3i	3	A 2 6 9	
3j	3	A 2 7 9	
3k	3	A 2 8 9	
4a	4	A B C 1	1, 2, 3, 4, 5
4b	4	A B 1 2	1
4c	4	A B 1 7	4
4d	4	A C 5	2, 7, 8
4e	4	A 3 5 7	
4f	4	A 5 7 8	
4g	4	2 5	7
4h	4	A 1 5 7	
4i	4	C 5 7 8	
4j	4	C 1 5 7	
4k	4	A C 9	12
4l	4	A 1 7 9	
4m	4	A 3 7 9	
4n	4	A 7 8 9	
4o	4	C 7 8 9	
4p	4	2 9	
5a	5	A B 1 2	1
5b	5	A B 1 7	4
5c	5	A B C 1 8	5
5d	5	A B C 1 6	3
5e	5	A 2 4	6
5f	5	A C 4 7	10
5g	5	A C 4 6	9
5h	5	A 1 4 7	
5i	5	A C 4 8	11
5j	5	A 3 4 7	10
5k	5	9	12

Theorem	Postulate	follows from	Set in which used
6a	6	A B C 2	1, 6, 7
6b	6	A B 2 7	
6c	6	1 7	4
6d	6	A 3 7	10
6e	6	1 8	5
6f	6	A 3 8	11
6g	6	A B 2 8	
6h	6	A B C 3 5	8
6i	6	A C 8	5, 11
6j	6	A B C 1 5	2
6k	6	A B C 9	12
6l	6	A C 7 9	
6m	6	A 2 7 9	
6n	6	A 2 8 9	
6o	6	A 4 8 9	
6p	6	A 7 8 9	
6q	6	C 8 9	
7a	7	A B C 2	1, 6, 7
7b	7	A B C 6	3, 9
7c	7	A C 4 6	9
7d	7	2 6	
7e	7	2 8	
7f	7	A C 8	5, 11
7g	7	A B C 5	2, 7, 8
7h	7	A C 5 6	
7i	7	A 4 5 8	
7j	7	A 1 4 5 6	
7k	7	A B C 9	12
7l	7	A C 6 9	
7m	7	A 4 8 9	
7n	7	A 1 4 6 9	
8a	8	A B C 2	1, 6, 7
8b	8	A B C 1 5	2
8c	8	A B C 3 5	8
8d	8	A B C 3 6	9
8e	8	1 7	4
8f	8	A B C 1 6	3
8g	8	A 3 7	10
8h	8	2 6	
8i	8	A C 5 6	
8j	8	A C 4 6	9
8k	8	A B 2 7	
8l	8	A B 1 5 6	
8m	8	A 1 4 5 6	
8n	8	A B C 9	12
8o	8	A B 1 6 9	
8p	8	A C 6 9	
8q	8	A C 7 9	
8r	8	A 1 4 6 9	
8s	8	A 2 7 9	
9a	9	A B 1 2	1
9b	9	A B C 1 5	2
9c	9	A B C 1 6	3
9d	9	A B C 1 7	4
9e	9	A B C 1 8	5
9f	9	A B C 2 4	6
9g	9	A B C 2 5	7
9h	9	A B 3 5	8
9i	9	A B C 3 4 6	9
9j	9	A B 3 4 7	10
9k	9	A B C 3 4 8	11
9l	9	A B 2 3 4	
9m	9	A B 1 3 7	

Examples of pseudo-betweenness.

In order to prove that no other theorems on deducibility are possible except those stated above, we first exhibit 54 examples of pseudo-betweenness, that is, 54 examples of systems K, R, which have some but not all of the properties mentioned in our basic list.

Of these examples, 37 were given in the earlier paper, and 17 are new. In the table following, the numbering of the examples is so arranged as to avoid conflict with the numbering in the earlier paper. (It will be noted that seven examples of the old list, namely, 17, 22, 25, 27, 31, 34, 35, are now omitted, being no longer needed, in view of certain of the new examples.)

In the case of each example, the postulates which are satisfied are mentioned explicitly, while the postulates which are not satisfied are indicated by a minus sign.

The new examples are as follows (the class K consisting of four elements, 1, 2, 3, 4, and the triads explicitly listed in each case being the only triads for which the relation R is supposed to be true):

Ex. 41. 123, 134, 142, 143, 213, 214, 234, 241, 312, 321, 324, 341, 412, 423, 431, 432.

Ex. 42. 123, 124, 142, 241, 243, 321, 324, 342, 421, 432.

Ex. 43. 123, 143, 214, 243, 314, 321, 324, 412, 413, 423.

Ex. 44. 123, 143, 214, 231, 243, 312, 314, 412, 423, 431.

Ex. 45. 123, 132, 134, 142, 231, 241, 243, 321, 324, 342, 423, 431.

Ex. 46. 123, 124, 132, 134, 142, 213, 214, 231, 234, 241, 243, 312, 321, 324, 342, 412, 421, 423, 431, 432.

Ex. 47. 123, 142, 312, 314, 341, 342, 412, 423.

Ex. 48. 123, 321.

Ex. 49. 123, 142, 324, 341.

Ex. 50. 123, 124, 312, 412, 431, 432.

Ex. 51. 123, 124, 231, 234, 241, 243, 341.

Ex. 52. 123, 231, 312, 412, 423, 431.

Ex. 53. 123, 134, 421, 423.

Ex. 54. 123, 124, 132, 134, 143, 213, 214, 231, 243, 312, 321, 324, 341, 342, 412, 421, 423, 431.

Ex. 55. 123, 132, 142, 143, 213, 231, 241, 243, 312, 321, 341, 342.

Ex. 56. 123, 143, 214, 243, 321, 324, 341, 342, 412, 423.

Ex. 57. 123, 124, 143, 243, 312, 341, 342, 412, 423.

Lemmas on non-deducibility

We are now in position to prove 84 lemmas on non-deducibility, which, taken together, establish the fact that no other theorems on deducibility are possible besides the 116 theorems listed above.

TABLE II′. LIST OF 54 EXAMPLES OF PSEUDO-BETWEENNESS

Ex.	has properties													Lemma in which example is used
A	—	B	C	D	1	2	3	4	5	6	7	8	9	A. 1
B	A	—	C	D	1	2	3	4	5	6	7	8	9	B. 1
C	A	B	—	D	1	2	3	4	5	6	7	8	9	C. 1
D	A	B	C	—	1	2	3	4	5	6	7	8	9	D. 1
1	A	B	C	D	—	2	3	—	—	6	7	8	—	1.1, 4.1, 5.2, 9.2
2	A	B	C	D	—	—	—	4	5	6	7	8	—	1.2, 2.2, 3.1, 9.3
3	A	B	C	D	1	—	3	4	—	—	—	—	—	2.1, 5.1, 6.1, 7.1, 8.1, 9.1
4	A	B	C	D	—	—	—	4	5	—	7	—	—	6.2, 8.3
5	A	B	C	D	—	—	—	—	—	6	7	—	—	8.2
6	A	—	C	D	—	2	3	4	5	6	7	8	—	1.4
7	A	—	C	D	1	—	3	—	—	6	—	—	—	2.4, 7.2, 8.6
8	A	—	C	D	1	—	—	4	5	6	7	8	—	2.5, 3.5
9	A	—	C	D	—	2	—	4	5	—	7	—	—	3.6, 6.8, 8.5
10	A	—	C	D	1	2	3	—	—	6	7	8	—	4.4, 5.5
11	A	—	C	D	1	2	3	4	5	—	—	—	9	6.7, 7.3, 8.4
12	A	B	—	D	—	2	3	4	5	6	7	8	9	1.3
13	A	B	—	D	1	—	3	4	5	6	7	8	9	2.3
14	A	B	—	D	1	—	—	4	5	6	7	8	—	3.2, 9.5
15	A	B	—	D	—	2	—	4	5	6	7	8	—	3.3, 9.6
16	A	B	—	D	1	—	3	—	5	6	—	8	9	4.2, 7.4
18	A	B	—	D	1	—	3	4	—	6	—	8	—	5.3, 7.5
19	A	B	—	D	1	2	3	4	5	—	—	—	9	6.3, 7.6, 8.7
20	A	B	—	D	—	—	3	4	5	6	—	—	9	7.7, 8.8
21	A	B	—	D	—	—	—	4	—	6	7	8	—	5.4
23	A	B	—	D	1	—	3	4	—	6	—	—	—	8.10
24	A	B	—	D	—	—	—	4	5	—	7	8	—	6.4
26	—	B	C	D	1	—	3	4	5	6	7	8	9	2.6
28	—	B	C	D	1	2	3	—	—	6	7	8	—	4.5
29	—	B	C	D	1	—	3	—	5	6	—	8	9	4.6
30	—	B	C	D	1	—	3	4	5	6	—	8	9	7.8
32	—	B	C	D	—	2	3	4	5	—	7	8	—	6.11
33	—	B	C	D	1	2	3	4	—	6	7	8	—	5.6
36	—	B	C	D	1	—	3	4	5	6	—	—	9	8.14
37	—	B	C	D	—	—	3	—	5	6	7	—	—	4.8
38	—	B	—	D	1	—	3	—	5	6	7	8	9	4.10
39	A	—	—	D	—	2	—	4	5	—	7	8	—	6.9
40	A	—	—	D	1	—	3	—	5	6	—	—	9	8.11
41	A	B	—	D	—	—	—	4	5	6	7	8	9	3.4
42	A	—	—	D	—	2	—	4	5	—	—	—	9	3.7
43	—	B	C	D	1	—	—	—	5	6	7	—	9	4.7
44	—	B	C	D	—	—	3	—	5	—	7	—	9	4.9
45	A	B	—	D	—	—	—	4	5	—	7	—	9	6.5
46	A	B	—	D	—	—	—	—	5	—	—	8	9	6.6
47	—	B	—	D	—	2	3	4	5	—	7	8	9	6.13
48	A	—	C	D	1	2	3	4	5	6	7	8	—	9.7
49	—	B	C	D	1	2	3	4	5	6	7	8	—	9.8
50	—	B	C	D	—	2	3	4	5	6	7	8	9	1.5
51	—	B	C	D	1	2	—	4	5	6	7	8	9	3.8
52	—	B	C	D	—	2	3	4	5	—	7	—	9	6.12, 8.12
53	—	B	C	D	1	2	3	4	5	—	—	—	9	6.10, 7.9, 8.15
54	A	B	—	D	—	—	—	4	5	6	7	—	9	8.9
55	A	B	—	D	1	—	3	4	—	6	—	8	—	9.4
56	A	B	—	D	—	—	—	—	5	6	7	—	9	4.3
57	—	B	C	D	—	—	3	4	5	6	7	—	9	8.13

TABLE III′. 84 LEMMAS ON NON-DEDUCIBILITY

Lemma	Postulate	is not deducible from	Proof by Ex.
A.1	A	B C D 1 2 3 4 5 6 7 8 9	A
B.1	B	A C D 1 2 3 4 5 6 7 8 9	B
C.1	C	A B D 1 2 3 4 5 6 7 8 9	C
D.1	D	A B C 1 2 3 4 5 6 7 8 9	D
1.1	1	A B C D 2 3 6 7 8	1
1.2	1	A B C D 4 5 6 7 8	2
1.3	1	A B D 2 3 4 5 6 7 8 9	12
1.4	1	A C D 2 3 4 5 6 7 8	6
1.5	1	B C D 2 3 4 5 6 7 8 9	50
2.1	2	A B C D 1 3 4	3
2.2	2	A B C D 4 5 6 7 8	2
2.3	2	A B D 1 3 4 5 6 7 8 9	13
2.4	2	A C D 1 3 6	7
2.5	2	A C D 1 4 5 6 7 8	8
2.6	2	B C D 1 3 4 5 6 7 8 9	26
3.1	3	A B C D 4 5 6 7 8	2
3.2	3	A B D 1 4 5 6 7 8	14
3.3	3	A B D 2 4 5 6 7 8	15
3.4	3	A B D 4 5 6 7 8 9	41
3.5	3	A C D 1 4 5 6 7 8	8
3.6	3	A C D 2 4 5 7	9
3.7	3	A D 2 4 5 9	42
3.8	3	B C D 1 2 4 5 6 7 8 9	51
4.1	4	A B C D 2 3 6 7 8	1
4.2	4	A B D 1 3 5 6 8 9	16
4.3	4	A B D 5 6 7 9	56
4.4	4	A C D 1 2 3 6 7 8	10
4.5	4	B C D 1 2 3 6 7 8	28
4.6	4	B C D 1 3 5 6 8 9	29
4.7	4	B C D 1 5 6 7 9	43
4.8	4	B C D 3 5 6 7	37
4.9	4	B C D 3 5 7 9	44
4.10	4	B D 1 3 5 6 7 8 9	38
5.1	5	A B C D 1 3 4	3
5.2	5	A B C D 2 3 6 7 8	1
5.3	5	A B D 1 3 4 6 8	18
5.4	5	A B D 4 6 7 8	21
5.5	5	A C D 1 2 3 6 7 8	10
5.6	5	B C D 1 2 3 4 6 7 8	33
6.1	6	A B C D 1 3 4	3
6.2	6	A B C D 4 5 7	4
6.3	6	A B D 1 2 3 4 5 9	19
6.4	6	A B D 4 5 7 8	24
6.5	6	A B D 4 5 7 9	15
6.6	6	A B D 5 8 9	46
6.7	6	A C D 1 2 3 4 5 9	11
6.8	6	A C D 2 4 5 7	9
6.9	6	A D 2 4 5 7 8	39
6.10	6	B C D 1 2 3 4 5 9	53
6.11	6	B C D 2 3 4 5 7 8	32
6.12	6	B C D 2 3 4 5 7 9	52
6.13	6	B D 2 3 4 5 7 8 9	47
7.1	7	A B C D 1 3 4	3
7.2	7	A C D 1 3 6	7
7.3	7	A C D 1 2 3 4 5 9	11
7.4	7	A B D 1 3 5 6 8 9	16
7.5	7	A B D 1 3 4 6 8	18
7.6	7	A B D 1 2 3 4 5 9	19
7.7	7	A B D 3 4 5 6 9	20
7.8	7	B C D 1 3 4 5 6 8 9	30
7.9	7	B C D 1 2 3 4 5 9	53
8.1	8	A B C D 1 3 4	3
8.2	8	A B C D 6 7	5
8.3	8	A B C D 4 5 7	4
8.4	8	A C D 1 2 3 4 5 9	11
8.5	8	A C D 2 4 5 7	9
8.6	8	A C D 1 3 6	7
8.7	8	A B D 1 2 3 4 5 9	19
8.8	8	A B D 3 4 5 6 9	20
8.9	8	A B D 4 5 6 7 9	54
8.10	8	A B D 1 3 4 6	23
8.11	8	A D 1 3 5 6 9	40
8.12	8	B C D 2 3 4 5 7 9	52
8.13	8	B C D 3 4 5 6 7 9	57
8.14	8	B C D 1 3 4 5 6 9	36
8.15	8	B C D 1 2 3 4 5 9	53
9.1	9	A B C D 1 3 4	3
9.2	9	A B C D 2 3 6 7 8	1
9.3	9	A B C D 4 5 6 7 8	2
9.4	9	A B D 1 3 4 6 8	55
9.5	9	A B D 1 4 5 6 7 8	14
9.6	9	A B D 2 4 5 6 7 8	15
9.7	9	A C D 1 2 3 4 5 6 7 8	48
9.8	9	B C D 1 2 3 4 5 6 7 8	49

Many of these lemmas were given in the earlier paper; but the new lemmas made necessary by the introduction of postulate 9 so often include certain of the old lemmas, that it is convenient to write out the whole list afresh, using a decimal notation instead of the letters of the alphabet, to avoid all possible confusion. This is done in Table III′, above.

It will be noticed that postulate D plays a peculiar rôle. Although it is strictly independent and therefore cannot be omitted, yet it is not used in proving any of the theorems on deducibility, and it may always be made to hold or fail without affecting the holding or failing of any other postulate. It may therefore be called not only independent but altogether "*detached*".

Complete independence of postulates A, B, C, D, 9

To establish the complete independence* of the five postulates A, B, C, D, 9, we exhibit $2^5 = 32$ examples, which we number 000—031 inclusive, in Table IV. In this table, a plus sign (+) indicates that a postulate is satisfied, a minus sign (—) that it fails.

Table IV. List of 32 examples for postulates A, B, C, D, 9

Ex.	A	B	C	D	9
000	+	+	+	+	+
001	−	+	+	+	+
002	+	−	+	+	+
003	+	+	−	+	+
004	+	+	+	−	+
005	+	+	+	+	−
006	−	−	+	+	+
007	−	+	−	+	+
008	−	+	+	−	+
009	−	+	+	+	−
010	+	−	−	+	+
011	+	−	+	−	+
012	+	−	+	+	−
013	+	+	−	−	+
014	+	+	−	+	−
015	+	+	+	−	−

Ex.	A	B	C	D	9
016	−	−	−	+	+
017	−	−	+	−	+
018	−	−	+	+	−
019	−	+	−	−	+
020	−	+	−	+	−
021	−	+	+	−	−
022	+	−	−	−	+
023	+	−	−	+	−
024	+	−	+	−	−
025	+	+	−	−	−
026	−	−	−	−	+
027	−	−	−	+	−
028	−	−	+	−	−
029	−	+	−	−	−
030	+	−	−	−	−
031	−	−	−	−	−

Example 000 shows that the five postulates are *consistent*.

Examples 001—005 show that the five postulates are *independent* in the ordinary sense; that is, no one of them is deducible from the other four.

* E. H. Moore, *Introduction to a form of general analysis, New Haven Colloquium,* 1906, published by the Yale University Press, New Haven, 1910; p. 82.

Examples 001—005 may be called "near-betweenness" systems, since they possess all but one of the five properties of betweenness. Examples 006—015 fail on two postulates; examples 016—025 fail on three, and examples 026—030 on four; while example 031 fails to have any one of the properties characteristic of betweenness.

Ex. 000. 123, 124, 134, 234, 321, 421, 431, 432.

Ex. 001. 123, 124, 134, 234.
Ex. 002. 123, 124, 321, 421.
Ex. 003. 123, 124, 134, 234, 321, 324, 421, 423, 431, 432.
Ex. 004. 123, 124, 134, 234, 321, 421, 431, 432; 444.
Ex. 005. 123, 143, 214, 234, 321, 341, 412, 432.

Ex. 006. 123, 124.
Ex. 007. 123, 124, 324, 341, 342.
Ex. 008. 123, 124, 134, 234; 444.
Ex. 009. 123, 142, 324, 341.
Ex. 010. 123, 124, 142, 143, 241, 321, 341, 421.
Ex. 011. 123, 124, 321, 421; 444.
Ex. 012. 123, 234, 321, 432.
Ex. 013. 123, 124, 134, 234, 321, 324, 421, 423, 431, 432; 444.
Ex. 014. 123, 214, 243, 314, 321, 324, 342, 412, 413, 423.
Ex. 015. 123, 143, 214, 234, 321, 341, 412, 432; 444.

Ex. 016. 123, 124, 132, 134.
Ex. 018. 123, 241.
Ex. 020. 123, 124, 134, 234, 243.
Ex. 023. 123, 124, 142, 241, 321, 421.
Exs. 017, 019, 021, 022, 024, 025. Same as Exs. 006, 007, 009, 010, 012, 014, with 444 added.

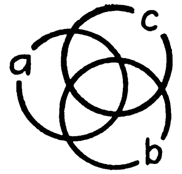
Ex. 026. 123, 124, 132, 134; 444.
Ex. 027. 123, 124, 132.
Exs. 028, 029, 030. Same as Exs. 018, 020, 023, with 444 added.

Ex. 031. 123, 213, 234, 243, 423; 444.

This last system (Ex. 031) will be found to violate all the thirteen postulates of our basic list; it is therefore as far removed as possible from a true betweenness system.

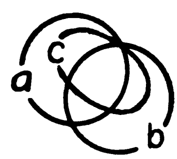
SIGNIFICANCE OF THE NOTION OF COMPLETE INDEPENDENCE*

The significance of the notion of complete independence derives from the fact that every postulate may be stated, at pleasure, in either the positive or the negative form, so that every postulate, a, should be regarded as a pair of coördinate propositions, a and $\bar{a}$. Thus a set of three postulates, $(a, \bar{a})$, $(b, \bar{b})$, $(c, \bar{c})$, divides the universe of discourse into $2^3 = 8$ compartments, represented by the logical products, abc; $\bar{a}bc$, $a\bar{b}c$, $ab\bar{c}$; $\bar{a}\bar{b}c$, $\bar{a}b\bar{c}$, $a\bar{b}\bar{c}$; $\bar{a}\bar{b}\bar{c}$, in which the barred and unbarred letters play precisely coördinate rôles.



If now there is no special relation between the postulates, all these compartments will be actually represented in the universe; it is only in the special case when some relation of implication among the propositions a, $\bar{a}$, b, $\bar{b}$, c, $\bar{c}$ holds true, that any one of these compartments will be empty.

For example, if $\bar{a}\bar{b}c$ is empty, then $\bar{a}c$ implies b (and also $\bar{b}c$ implies a, and $\bar{a}\bar{b}$ implies $\bar{c}$); and, conversely, if any one of these three implications is valid, then the compartment $\bar{a}\bar{b}c$ will be empty. Similarly for each of the other compartments.



Hence Moore's criterion is a natural one: a set of n postulates is "completely independent" when and only when no one of the 2^n compartments into which the postulates divide the universe is empty.

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* Among the many papers on "complete independence" which have appeared in recent years may be mentioned the following:

R. D. Beetle, *On the complete independence of Schimack's postulates for the arithmetic mean,* Mathematische Annalen, vol. 76 (1915), pp. 444–446;

L. L. Dines, *Complete existential theory of Sheffer's postulates for Boolean algebras,* Bulletin of the American Mathematical Society, vol. 21 (1915), pp. 183–188;

E. V. Huntington, *Complete existential theory of the postulates for serial order*; and *Complete existential theory of the postulates for well ordered sets,* Bulletin of the American Mathematical Society, vol. 23 (1917), pp. 276–280 and pp. 280–282;

J. S. Taylor, *Complete existential theory of Bernstein's set of four postulates for Boolean algebras,* Annals of Mathematics, ser. 2, vol. 19 (1917), pp. 64–69; and *Sheffer's set of five postulates for Boolean algebras in terms of the operation "rejection" made completely independent,* Bulletin of the American Mathematical Society, vol. 26 (1920), pp. 449–454;

B. A. Bernstein, *On the complete independence of Hurwitz's postulates for abelian groups and fields,* Annals of Mathematics, ser. 2, vol. 23 (1922), pp. 313-316; and *The complete existential theory of Hurwitz's postulates for abelian groups and fields,* Bulletin of the American Mathematical Society, vol. 28 (1922), pp. 397-399, and vol. 29 (1923), p. 33;

E. V. Huntington, *Sets of completely independent postulates for cyclic order,* Proceedings of the National Academy of Sciences, February, 1924;

W. E. Van de Walle, *On the complete independence of the postulates for betweenness,* in the present number of these Transactions.

Appendix, on the relation of betweenness to cyclic order

The theory of *betweenness* (that is, the order of points along a straight line, without distinction of sense along the line), is closely related to the theory of *cyclic order* (that is, the order of points on a closed curve with a definite sense around the curve).*

Betweenness is characterized by the completely independent postulates A, B, C, D, 9; cyclic order† by the completely independent postulates E, B, C, D, 9.

The postulates B, C, D, 9 hold true in both theories, while postulates A and E differ only by the interchange of two letters; thus:

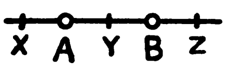
Postulate A (*for betweenness*). *If ABC, then CBA.*

Postulate E (*for cyclic order*). *If ABC, then CAB.*

The following theorems may serve to bring out the contrast between the two theories.

Theorem on betweenness. (From A, C, 9.) *If A, B are two distinct elements, and if X, Y, Z are three other distinct elements, distinct from A and B, and such that XAB, AYB, ABZ; then XYZ.*

Proof. By 9, $XAB\,.\,Y\,.\supset .\,XAY\smile YAB$; and by 9, $ABZ\,.\,Y\,.\supset .\,ABY\smile YBZ$.



But YAB and ABY conflict with AYB, by C and A; hence XAY and YBZ. Again, by 9, $AYB\,.\,X\,.\supset .\,AYX\smile XYB$. But AYX conflicts with XAY, by C and A; hence XYB. Then by 9, $XYB\,.\,Z\,.\supset .\,XYZ\smile ZYB$.

But ZYB conflicts with YBZ, by C and A; hence XYZ.

Theorem on cyclic order. (From E, C, 9.) *If A, B, C are three distinct elements, such that ABC; and if X, Y, Z are three other distinct elements, distinct from A, B, C and such that AXB, BYC, CZA; then XYZ.*

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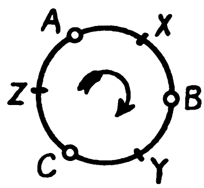
* Besides (1) *betweenness* and (2) *cyclic order*, both of which are expressed in terms of a triadic relation, there are two other important types of order, namely: (3) *serial order* (that is, the order of points along a straight line with a definite sense along the line), which is expressed in terms of a dyadic relation; and (4) *separation of point pairs* (that is, the order of points on a closed curve without distinction of sense around the curve), which is expressed in terms of a tetradic relation. Sets of completely independent postulates for serial order are well known (loc. cit.); similar sets for the separation of point pairs will form the subject of a later paper.

† For the set E, B, C, D, 9, and two equivalent sets, E, B, C, D, 2 and E, B, C, D, 3, see E. V. Huntington, *Sets of completely independent postulates for cyclic order,* Proceedings of the National Academy of Sciences, February, 1924.



Proof. By 9, $ABC.Y.\supset.ABY\smile YBC$, whence by E, $YAB\smile BCY$. But BCY conflicts with BYC, by C. Hence YAB. Then by 9, $YAB.X.\supset.YAX\smile XAB$. But XAB conflicts with AXB, by E and C. Hence YAX, whence by E, AXY.

By E and 9, $BCA.Y.\supset.BCY\smile YCA$. But BCY conflicts with BYC, by C. Hence YCA. Then by E and 9, $CAY.Z.\supset.CAZ\smile ZAY$. But CAZ conflicts with CZA, by C. Hence ZAY. Then by E and 9, $AYZ.X.\supset.AYX\smile XYZ$.

But AYX conflicts with AXY, by C. Hence XYZ.

The six postulates A, E, B, C, D, 9, taken together, would form, of course, an *inconsistent set,* since no system (K, R) has all these properties. It is interesting, however, to note the following "theorems of deducibility" among these six postulates.

THEOREM 201. *Proof of* 9 *from* A, E, B.

To prove: $ABC.X.\supset.ABX\smile XBC$. By B, at least one of the six permutations of A, B, X will be true; hence, by A and E, all six will be true, so that ABX will be true. Similarly, XBC will be true.

THEOREM 202. *Proof of* 9 *from* A, E, C.

To prove: $ABC.X.\supset.ABX\smile XBC$. Suppose 9 fails; that is, suppose ABX and XBC are both false while ABC is true. Then by A and E, we have CBA and CAB, which conflict with each other, by C. Hence 9 must hold.

THEOREM 203. *Proof of* B *from not*-A, E, C, *and* 9.

To prove: If A, B, C are distinct, then at least one of the six permutations, ABC, ACB, BAC, BCA, CAB, CBA, is true; or, more briefly: If A, B, C are distinct, then $P(A, B, C)$.*

Since postulate A is violated, there must exist at least one true triad, say XYZ.

Let A be any element distinct from X, Y, Z.

By 9, $XYZ.A.\supset.XYA\smile AYZ$. But if XYA, then by E and 9, $YAX.Z.\supset.YAZ\smile ZAX$; and if AYZ, then by E and 9, $ZAY.X.\supset.ZAX\smile XAY$.

Therefore $YAZ\smile ZAX\smile XAY$.

Case 1. If YAZ, then by E and 9, $AZY.X.\supset.AZX\smile XZY$; and by E and 9, $ZYA.X.\supset.ZYX\smile XYA$. But XZY and ZYX conflict with XYZ, by E and C; hence, in Case 1, AZX and XYA.

* For essential details of this proof, including the convenient notation $P(A, B, C)$, I am indebted to Mr. C. H. Langford.

Case 2. If ZAX, then by E and 9, $AXZ.Y. \supset .AXY \smile YXZ$; and by E and 9, $XZA.Y. \supset .XZY \smile YZA$. But YXZ and XZY conflict with XYZ, by E and C; hence, in Case 2, AXY and YZA.

Case 3. If XAY, then by E and 9, $AYX.Z. \supset .AYZ \smile ZYX$; and by E and 9, $YXA.Z. \supset .YXZ \smile ZXA$. But ZYX and YXZ conflict with XYZ, by E and C; hence, in Case 3, AYZ and ZXA.

Therefore (making use of E), we have

$$XYZ.A. \supset .(AZY.AZX.AXY) \smile (AXZ.AXY.AYZ) \smile (AYX.AYZ.AZX),$$

whence

$$P(X,Y,Z).A. \supset .P(A,Y,Z).P(A,Z,X).P(A,X,Y),$$

where the notation $P(A,X,Y)$, for example, means that at least one of the six possible permutations of the three letters, A, X, Y, forms a true triad.

Now let B be any element distinct from X, Y, Z, A. Then, by the same reasoning,

$$P(A,Y,Z).B. \supset .P(B,Y,Z).P(B,A,Z).P(B,A,Y).$$

Finally, let C be any element distinct from X, Y, Z, A, B. Then

$$P(B,A,Y).C. \supset .P(C,A,Y).P(C,B,Y).P(C,B,A).$$

This last result, $P(C,B,A)$, states that at least one of the permutations of the letters C, B, A forms a true triad, which establishes the theorem.

THEOREM 204. *Proof of not*-B *from* A, E, C.

To prove: that three elements, A, B, C, exist, such that all six permutations, ABC, ACB, BAC, BCA, CAB, CBA, are false.

If the system contains no true triad, then the theorem is clearly true. If the system contains any true triad, say XYZ, then by A and E, we have ZYX and ZXY, which is impossible, by C. Hence the theorem is true.

We are now prepared to exhibit the "complete existential theory" of these six postulates A, E, B, C, D, 9. The six postulates divide the universe into $2^6 = 64$ compartments, some of which, however, will be "empty." Thus, the four theorems just proved show that examples of the types

A, E, B, $\bar{9}$; A, E, C, $\bar{9}$; A, E, B, C; $\bar{A}$, E, $\bar{B}$, C, 9 are impossible, so that at least ten of the 64 compartments will be empty (see the list in Table V below). This list shows that all the remaining 54 examples actually exist, so that the "existential theory" is complete.

Table V. Examples for the six inconsistent postulates A, E, B, C, D, 9

Rec.	A	E	B	C	D	9	Ex.
(1)	+	+	+	+	+	+	—
(2)	+	+	+	+	+	—	—
3	+	+	+	—	+	+	037
(4)	+	+	+	—	+	—	—
5	+	+	—	+	+	+	038
(6)	+	+	—	+	+	—	—
7	+	+	—	—	+	+	039
8	+	+	—	—	+	—	040
9	+	—	+	+	+	+	000
10	+	—	+	+	+	—	005
11	+	—	+	—	+	+	003
12	+	—	+	—	+	—	014
13	+	—	—	+	+	+	002
14	+	—	—	+	+	—	012
15	+	—	—	—	+	+	010
16	+	—	—	—	+	—	023
17	—	+	+	+	+	+	033
18	—	+	+	+	+	—	034
19	—	+	+	—	+	+	035
20	—	+	+	—	+	—	036
(21)	—	+	—	+	+	+	—
22	—	+	—	+	+	—	041
23	—	+	—	—	+	+	043
24	—	+	—	—	+	—	042
25	—	—	+	+	+	+	001
26	—	—	+	+	+	—	009
27	—	—	+	—	+	+	007
28	—	—	+	—	+	—	020
29	—	—	—	+	+	+	006
30	—	—	—	+	+	—	018
31	—	—	—	—	+	+	016
32	—	—	—	—	+	—	027

Records 33–64 are the same as Records 1–32 with D + changed to D—, and the letter "d" added to each example-number (in so far as these numbers exist).

The requisite examples, not already listed under Table IV, are as follows:

Ex. 033. 123, 231, 312; 124, 241, 412; 134, 341, 413; 234, 342, 423.

Ex. 034. 123, 231, 312; 214, 142, 421; 134, 341, 413; 432, 324, 243.

Ex. 035. 123, 231, 312; 312, 213, 132; 421, 214, 142; 431, 314, 143; 234, 342, 423; 432, 324, 243.

Ex. 036. 123, 231, 312; 214, 142, 421; 134, 341, 413; 432, 324, 243; 321, 132, 213.

Ex. 037. All the twenty-four possible triads are true.

Ex. 038. No triads true.

Ex. 039. 123, 231, 312; 321, 132, 213; 124, 241, 412; 421, 214, 142; 234, 342, 423; 432, 324, 243.

Ex. 040. 123, 231, 312; 321, 213, 132; 124, 241, 412; 421, 214, 142.

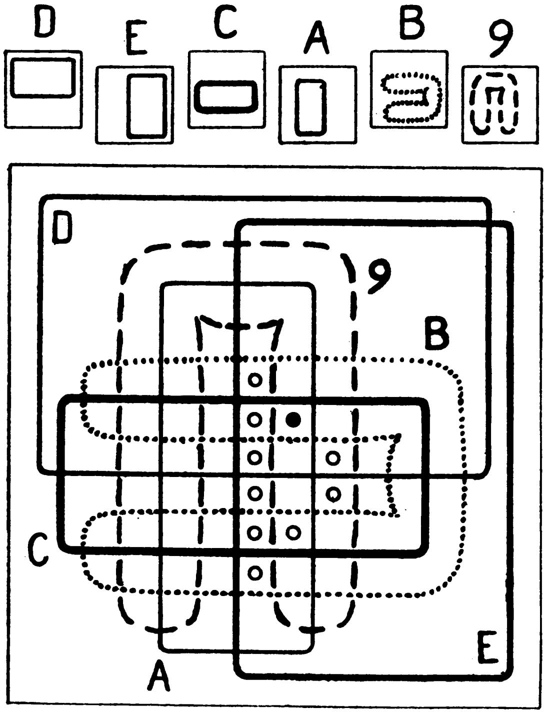
Ex. 041. 123, 231, 312; 214, 142, 421.

Ex. 042. 123, 231, 312; 321, 213, 132; 214, 142, 421.

Ex. 043 Here the class K consists of 5 elements, 1, 2, 3, 4, 5; all the sixty possible triads are true *except* the following: 321, 213, 132; 345, 453, 534; 543, 435, 354.

Exs.033d, 034d, etc., are the same as Exs. 033, 034, etc., with the addition of the triad 444 (so as to violate postulate D).

Finally, the inter-relations between the six postulates A, E, B, C, D, 9 may be shown diagrammatically as in the accompanying figure.* In this diagram, a zero in any compartment indicates that no example having the properties belonging to that compartment exists. For instance, the fact that no example of the type A, E, B, C, D, 9 exists, shows that the postulates are inconsistent.



* This form of diagram, the possibility of which was vaguely suggested by Venn in 1881, is believed to be an improvement over those in common use. See

John Venn, *On the diagrammatic and mechanical representation of propositions and reasonings,* Philosophical Magazine, ser. 5, vol. 10 (July, 1880), pp. 1–18; or his *Symbolic Logic,* 1st edition, 1881, p. 108, 2d edition, 1894, p. 118 (with extensive historical notes);

H. Marquand, *On logical diagrams for n terms,* Philosophical Magazine, ser. 5, vol. 12 (October, 1881), pp. 266–270;

C. L. Dodgson ["Lewis Carroll"], *Symbolic Logic,* London, 1896, known to me only through a citation by C. I. Lewis;

W. J. Newlin, *A new logical diagram,* Journal of Philosophy, Psychology, and Scientific Methods, vol. 3 (1906), pp. 539–545;

W. E. Hocking, *Two extensions of the use of graphs in elementary logic,* University of California Publications in Philosophy, vol. 2 (1909), pp. 31–44; and

C. I. Lewis, *A Survey of Symbolic Logic,* University of California Press, 1918, p. 180.

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